

Regularization by noise in (2x 2) hyperbolic systems of conservation law.

Christian Olivera*

Key words and phrases. Stochastic partial differential equation, Continuity equation, Hyperbolic Systems, Entropy solution, Regularization by noise.

MSC2010 subject classification: 60H15, 35R60, 35L02, 60H30, 35L40.

Abstract

In this paper we study a non strictly systems of conservation law by stochastic perturbation. We show the existence and uniqueness of the solution. We do not assume that BV - regularity for the initial conditions. The proofs are based in concept of entropy solution and in the characteristics method (in the influence of noise). This is the first result on the regularization by noise in hyperbolic systems of conservation law.

1 Introduction

A large number of problems in physics and engineering are modeled by systems of conservation laws

$$\partial_t u(t, x) + \operatorname{div}(F(u(t, x))) = 0, \quad (1.1)$$

here $u = u$ is called the conserved quantity, while F is the flux. Examples for hyperbolic systems of conservation laws include the shallow water equations of oceanography, the Euler equations of gas dynamics, the magnetohydrodynamics (MHD) equations of plasma physics, the equations of nonlinear elastodynamics and the Einstein equations of general relativity. When smooth

*Departamento de Matemática, Universidade Estadual de Campinas, Brazil. E-mail: colivera@ime.unicamp.br.

initial data, it is well known that the solution can develop shocks within finite time. Therefore, global solutions can only be constructed within a space of discontinuous functions. Moreover, when discontinuities are present, weak solutions may not be unique. A central issue is to regain uniqueness by imposing appropriate selection criteria. The well-posedness theorems within the class of entropy solutions, for the scalar case, were established by Kruzkov (see [18]). It is well known that the main techniques of abstract functional analysis do not apply hyperbolic systems. Solutions cannot be represented as fixed points of continuous transformations, or in variational form, as critical points of suitable functionals. For the above reasons, the theory of hyperbolic conservation laws has largely developed by ad hoc methods. We refer to [6], [5] [9] and [27]. The well-posedness general system of conservation laws has been established only for initial data with sufficiently small total variation, see [6] and the references therein.

We consider the following systems of conservation law

$$\begin{cases} \partial_t v(t, x) + \operatorname{Div}(f(v)) = 0 \\ \partial_t v(t, x) + \operatorname{Div}(vu) = 0. \end{cases} \quad (1.2)$$

We point that in the $L^1 \cap L^\infty$ setting this systems ill-posedness since the classical DiPerna-Lions-Ambrossio theory of uniqueness of distributional solutions for transport/continuity equation does not apply when the drift has $L^1 \cap L^2$ regularity, see [1] and [11]. Also see [2] and [10] for new developments in the theory.

In contrast with its deterministic counterpart, the singular stochastic continuity/transport equation with multiplicative noise is well-posed. The addition of a stochastic noise is often used to account for numerical, empirical or physical uncertainties. In [3, 4, 12, 13, 15, 24, 25], well-posedness and regularization by linear multiplicative noise for continuity/transport equations have been obtained. We refer to [25] for more details on the literature.

In this paper we study the influence of the noise in the hyperbolic systems (1.2). More precisely, we consider following stochastic systems of conservation law

$$\begin{cases} \partial_t v(t, x) + \text{Div}((F(v(t, x))) = 0, \\ \partial_t u(t, x) + \text{Div}\left((u + \frac{dB_t}{dt}) \cdot u(t, x)\right) = 0, \\ v|_{t=0} = v_0, \quad u|_{t=0} = u_0. \end{cases} \quad (1.3)$$

Here, $(t, x) \in [0, T] \times \mathbb{R}$, $\omega \in \Omega$ is an element of the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and B_t is a standard Brownian motion in \mathbb{R} . The stochastic integration is to be understood in the Stratonovich sense. The Stratonovich form is the natural one for several reasons, including physical intuition related to the Wong-Zakai principle.

The main issue of this paper is to prove existence and uniqueness of entropy-weak solutions for the stochastic systems of the conservation law (1.3). We do not assume BV -regularity for the initial conditions. We use the entropy formulation of conservation law and we employ the stochastic characteristics in order to obtain a unique solution to the one-dimensional stochastic equation with a bounded measurable drift coefficient. We adapted the ideas in [25] and [26] in our context where the drift term in the continuity equation depend on time and it is bounded and integrable.

Throughout of this paper, we fix a stochastic basis with a d -dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$.

1.1 One Example

We consider the systems

$$\begin{cases} \partial_t v(t, x) + \text{Div}\left(\frac{1}{2}v^2(t, x)\right) = 0, \\ \partial_t u(t, x) + \text{Div}(v(t, x)u(t, x)) = 0, \\ v|_{t=0} = v_0, \quad u|_{t=0} = u_0, \end{cases} \quad (1.4)$$

here v is the velocity and u the density of the particles. This system has applications in cosmology, the model describes the evolution of matter in the last stage of the expansion of the universe as cold dust moving under gravity alone and the laws are governed by the system (1.4). Clearly the eigenvalues are equal $\lambda_1 = \lambda_2 = v$. Thus the system (1.4) is not strictly hyperbolic. The first equation of (1.4)-Burgers equation is known to develop

singularities in finite time even if the initial data v_0 is smooth, and it is not at all obvious to solve the second equation. One question that remains is a well-posedness theory and large time behaviour of solution. In [17] and [28] the authors proved existence of weak solutions via δ -shock for Riemann initial condition. .

1.2 Scalar case.

We point that recently there has been an interest in studying the effect of stochastic forcing on nonlinear conservation laws driven by space-time white noise, see [7, 8, 14, 16]. For other hand, in [22] and [23] the authors introduced the theory of pathwise solutions to study the stochastic conservation law driven by continuous noise.

1.3 Hypothesis

We assume the following conditions

Hypothesis 1.1. *The flux F satisfies*

$$F \in C^1 \tag{1.5}$$

and the initial condition holds

$$v_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \tag{1.6}$$

2 Existence

2.1 Definition of solutions

Definition 2.1. *Let $\eta \in C^1(\mathbb{R})$ be a convex function. If there exist $q \in C^\mathbb{R}$ such that for all v*

$$\eta'(v)F'(v) = q'(v)$$

then η, q is called an entropy-entropy flux pair of the conservation law

$$\partial_t v(t, x) + \text{Div}(f(v)) = 0, \quad v(t, 0) = v_0(x).$$

Definition 2.2. *The stochastic process $v \in L^\infty([0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^1(\mathbb{R}))$ and $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1([0, T] \times \Omega \times \mathbb{R})$ are called a entropy weak solution of the stochastic hyperbolic systems (1.3) when:*

- v is entropy solution of the conservation law

$$\partial_t v(t, x) + \text{Div}(F(v)) = 0, \quad v(t, 0) = v_0(x).$$

That is, if for every entropy flux pair η, q we have

$$\partial_t \eta(v) + \text{Div}(q(v)) \leq 0$$

in the sense of distribution.

- For any $\varphi \in C_0^\infty(\mathbb{R})$, the real valued process $\int u(t, x) \varphi(x) dx$ has a continuous modification which is an \mathcal{F}_t -semimartingale, and for all $t \in [0, T]$, we have \mathbb{P} -almost surely

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) v(t, x) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx \odot dB_s. \end{aligned} \tag{2.7}$$

Remark 2.3. Using the same idea as in Lemma 13 [15], one can write the problem (2.7) in Itô form as follows, a stochastic process $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1([0, T] \times \Omega \times \mathbb{R})$ is solution of the SPDE (2.7) iff for every test function $\varphi \in C_0^\infty(\mathbb{R})$, the process $\int u(t, x) \varphi(x) dx$ has a continuous modification which is a \mathcal{F}_t -semimartingale and satisfies the following Itô's formulation

$$\begin{aligned} \int_{\mathbb{R}} u(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) v(t, x) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

2.2 Existence.

We shall prove existence of solutions under hypothesis 1.1.

Lemma 2.4. Assume that hypothesis 1.1 holds. Then there exists entropy-weak solution of the hyperbolic systems (1.3).

Proof. Step 1: Conservation law . According to the classical theory of conservation law, see for instance [9], we have that there exist a uniqueness entropy solution of the conservation law

$$\partial_t v(t, x) + \text{Div}(F(v)) = 0, \quad v(t, 0) = v_0(x).$$

If the the initial condition $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then the solution $v \in L^\infty([0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^1(\mathbb{R}))$.

Step 2: Primitive of v . It easy to see that for any test function $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} v(t, x) \varphi(x) dx = \int_{\mathbb{R}} v_0(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} F(v(s, x)) \partial_x \varphi(x) dx ds.$$

Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers. Then we obtain

$$\int_{\mathbb{R}} v(t, y) \rho_\varepsilon(x - y) dy = \int_{\mathbb{R}} v_0(y) \rho_\varepsilon(x - y) dy + \int_0^t \int_{\mathbb{R}} F(v(s, y)) \partial_y \rho_\varepsilon(x - y) dy ds.$$

Integrating we get

$$\int_0^z v_\varepsilon(t, x) dx = \int_0^z v_0^\varepsilon(x) dz + \int_0^t (F(v) * \rho_\varepsilon)(z) ds.$$

We denoted $\int_0^z v_\varepsilon(t, x) dx = \bar{v}_\varepsilon(t, x)$.

Step 3: Regularization. We define the family of regularized coefficients given by

$$v^\varepsilon(t, \cdot) = (v(t, x) * \rho_\varepsilon)(t, \cdot).$$

Clearly we observe that, for every $\varepsilon > 0$, any element v^ε , u_0^ε are smooth (in space) and with bounded derivatives of all orders. We observe that to study the stochastic continuity equation (SCE) (2.7) is equivalent to study the stochastic transport equation given by (regularized version):

$$\begin{cases} du^\varepsilon(t, x) + \nabla u^\varepsilon(t, x) \cdot (v^\varepsilon(t, x) dt + \circ dB_t) + \text{div} b^\varepsilon(x) u^\varepsilon(t, x) dt = 0, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon \end{cases} \quad (2.8)$$

Following the classical theory of H. Kunita [19, Theorem 6.1.9] we obtain that

$$u^\varepsilon(t, x) = u_0^\varepsilon(\psi_t^\varepsilon(t, x))J\psi_t^\varepsilon(t, x),$$

is the unique solution to the regularized equation (2.8), where ϕ_t^ε is the flow associated to the following stochastic differential equation (SDE):

$$dX_t = v^\varepsilon(t, X_t) dt + dB_t, \quad X_0 = x,$$

and ψ_t^ε is the inverse of ϕ_t^ε .

Step 4: Itô Formula . Applying the Itô formula to $\bar{v}_\varepsilon(t, X_t^\varepsilon)$ we deduce

$$\begin{aligned} \bar{v}_\varepsilon(t, X_t^\varepsilon) &= \int_0^{X_t^\varepsilon} u_0^\varepsilon(x) dx + \int_0^t (F(v) * \rho_\varepsilon)(s, X_s^\varepsilon) ds + \int_0^t v_\varepsilon^2(s, X_s^\varepsilon) ds \\ &\quad + \int_0^t v_\varepsilon(s, X_s^\varepsilon) dB_s + \frac{1}{2} \int_0^t (\partial_x v_\varepsilon)(s, X_s^\varepsilon) ds \end{aligned}$$

Step 5: Boundedness. We observe that

$$\begin{aligned} \|\bar{v}_\varepsilon(t, X_t^\varepsilon)\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} &\leq \|v\|_{L^\infty([0, T], L^1(\mathbb{R}))}, \\ \left\| \int_0^{X_t^\varepsilon} u_0^\varepsilon(x) dx \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} &\leq \|v_0\|_{L^1(\mathbb{R})}, \\ \left\| \int_0^t (F(v) * \rho_\varepsilon)(s, X_s^\varepsilon) ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} &\leq C \|F(v)\|_{L^\infty}, \\ \left\| \int_0^t v_\varepsilon^2(s, X_s^\varepsilon) ds \right\|_{L^\infty(\Omega \times [0, T] \times \mathbb{R})} C &\leq \|v\|_{L^2([0, T], L^\infty(\mathbb{R}))}, \end{aligned}$$

Step 6 : Estimation on Jacobain.

We denote

$$\mathcal{E} \left(\int_0^t v_\varepsilon(s, X_s) dB_s \right) = \exp \left\{ \int_0^t v_\varepsilon(s, X_s^\varepsilon) dB_s - \frac{1}{2} \int_0^t v_\varepsilon^2(s, X_s^\varepsilon) ds \right\},$$

We note that $\partial_x X_t$ satisfies

$$\partial_x X_t = \exp \left\{ \int_0^t (\partial_x v_\epsilon)(s, X_s) ds \right\}.$$

From steps 4-5 we have

$$\mathbb{E}|\partial_x X_t|^{-1} \leq C \mathbb{E} \left(\int_0^t v_\epsilon(s, X_s) dB_s \right).$$

We observe that the processes $\mathcal{E} \left(\int_0^t v_\epsilon(s, X_s) dB_s \right)$, is martingale with expectation equal to one. The we conclude that

$$\mathbb{E}|\partial_x X_t|^{-1} \leq C.$$

Step 7: Passing to the limit .

Making the change of variables $y = \psi_t^\epsilon(x)$ we have that

$$\int_{\mathbb{R}} \mathbb{E}[|u^\epsilon(t, x)|^2] dx = \int_{\mathbb{R}} |u_0^\epsilon(y)|^2 \mathbb{E}|J\phi_t^\epsilon|^{-1} dy.$$

From last step we have

$$\int_{\mathbb{R}} \mathbb{E}[|u^\epsilon(t, x)|^2] dx \leq C. \quad (2.9)$$

Therefore, the sequence $\{u^\epsilon\}_{\epsilon>0}$ is bounded in $L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1([0, T] \times \Omega \times \mathbb{R})$. Then there exists a convergent subsequence, which we denote also by u^ϵ , such that converge weakly in $L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$ to some process $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1([0, T] \times \Omega \times \mathbb{R})$.

Now, if u^ϵ is a solution of (2.8), it is also a weak solution, that is, for any test function $\varphi \in C_0^\infty(\mathbb{R})$, u^ϵ satisfies (written in the Itô form):

$$\begin{aligned} \int_{\mathbb{R}} u^\epsilon(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0^\epsilon(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u^\epsilon(s, x) v^\epsilon(s, x) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u^\epsilon(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^\epsilon(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

Thus, for prove existence of the SCE (1.3) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [15, theorem 15].

□

3 Uniqueness.

In this section, we shall present a uniqueness theorem for the SPDE (1.3)

Theorem 3.1. *Under the conditions of hypothesis 1.1, uniqueness holds for entropy -weak solutions of the hyperbolic problem (1.3).*

Proof. Step 1: Set of solutions. The uniqueness of the conservation law

$$\partial_t v(t, x) + \text{Div}(F(v)) = 0, \quad v(t, 0) = v_0(x).$$

we follow the classical theory of entropy solutions.

Step 2: We remark that the set of solutions of equation (2.7) is a linear subspace of $L^\infty([0, T] \times R, L^2(\Omega)) \cap L^1([0, T] \times \Omega \times \mathbb{R})$, because the stochastic continuity equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a u with initial condition $u_0 = 0$ vanishes identically.

Step 1: Primitive of the solution. We define $V(t, x) = \int_{-\infty}^x u(t, y) dy$. We consider a nonnegative smooth cut-off function η supported on the ball of radius 2 and such that $\eta = 1$ on the ball of radius 1. For any $R > 0$, we introduce the rescaled functions $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$. Let be $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} V(t, x) \varphi(x) \eta_R(x) dx = - \int_{\mathbb{R}} u(t, x) \theta(x) \eta_R(x) dx - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx,$$

where $\theta(x) = \int_{-\infty}^x \varphi(y) dy$. By definition of the solution u , taking as test function $\theta(x) \eta_R(x)$ we deduce that

$$\begin{aligned} \int_{\mathbb{R}} V(t, x) \eta_R(x) \varphi(x) dx &= - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) v(s, x) \eta_R(x) \varphi(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \eta_R(x) \varphi(x) dx \circ dB_s - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) v(s, x) \partial_x \eta_R(x) \theta(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \circ dB_s - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx. \end{aligned} \tag{3.10}$$

Taking the limit as $R \rightarrow \infty$ we get

$$\begin{aligned} & \int_{\mathbb{R}} V(t, x) \varphi(x) dx = \\ & - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) v(s, x) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \varphi(x) dx \circ dB_s. \end{aligned} \quad (3.11)$$

Step 2: Smoothing. Let $\{\rho_\varepsilon(x)\}_\varepsilon$ be a family of standard symmetric mollifiers. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we use $\rho_\varepsilon(x - \cdot)$ as test function and we obtain

$$\begin{aligned} \int_{\mathbb{R}} V(t, y) \rho_\varepsilon(x - y) dy &= - \int_0^t \int_{\mathbb{R}} (v(s, y) \partial_y V(s, y)) \rho_\varepsilon(x - y) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_y V(s, y) \rho_\varepsilon(x - y) dy \circ dB_s \end{aligned}$$

We put $V_\varepsilon(t, x) = (V * \rho_\varepsilon)(x)$, $v_\varepsilon(t, x) = (v * \rho_\varepsilon)(t, x)$ and $(vV)_\varepsilon(t, x) = (v.V * \rho_\varepsilon)(x)$. Then have

$$\begin{aligned} V_\varepsilon(t, x) &+ \int_0^t v_\varepsilon(s, x) \partial_x V_\varepsilon(s, x) ds + \int_0^t \partial_x V_\varepsilon(s, x) \circ dB_s \\ &= \int_0^t (\mathcal{R}_\varepsilon(V, v))(x, s) ds, \end{aligned}$$

where $\mathcal{R}_\varepsilon(V, v) = v_\varepsilon \partial_x V_\varepsilon - (v \partial_x V)_\varepsilon$.

Step 3: Method of Characteristic.

We consider the stochastic flow

$$dX_t^\varepsilon = v^\varepsilon(t, X_t^\varepsilon) dt + dB_t, \quad X_0 = x.$$

Using the same arguments that in steps 3-5-6 of the existence proof we have

$$\mathbb{E}|JX_{t-s}^\varepsilon|^2 \leq C. \quad (3.12)$$

Applying the Itô-Wentzell-Kunita formula to $V_\varepsilon(t, X_t^\varepsilon)$, see Theorem 8.3 of [20], we have

$$V_\varepsilon(t, X_t^\varepsilon) = \int_0^t (\mathcal{R}_\varepsilon(V, v))(X_s^\varepsilon, s) ds.$$

Hence

$$V_\varepsilon(t, x) = \int_0^t (\mathcal{R}_\varepsilon(V, v))(Y_{t-s}^\varepsilon, s) ds.$$

Multiplying by the test functions φ and integrating in \mathbb{R} we obtain

$$\int V_\varepsilon(t, x) \varphi(x) dx = \int_0^t \int (\mathcal{R}_\varepsilon(V, v))(Y_{t-s}^\varepsilon, s) \varphi(x) dx ds. \quad (3.13)$$

Doing the change of variable we obtain

$$\int_0^t \int (\mathcal{R}_\varepsilon(V, v))(Y_{t-s}^\varepsilon, s) \varphi(x) dx ds = \int_0^t \int (\mathcal{R}_\varepsilon(V, v))(x, s) JX_{t-s}^\varepsilon \varphi(X_{t-s}^\varepsilon) dx ds. \quad (3.14)$$

Step 4: Convergence of the commutator. Now, we observe that $\mathcal{R}_\varepsilon(V, b)$ converge to zero in $L^2([0, T] \times \mathbb{R})$. In fact, we have

$$(v \partial_x V)_\varepsilon \rightarrow v \partial_x V \text{ in } L^2([0, T] \times \mathbb{R}),$$

and by the dominated convergence theorem we obtain

$$v_\varepsilon \partial_x V_\varepsilon \rightarrow v \partial_x V \text{ in } L^2([0, T] \times \mathbb{R}).$$

Step 5: Conclusion. From step 3 we have

$$\int V_\varepsilon(t, x) \varphi(x) dx = \int_0^t \int (\mathcal{R}_\varepsilon(V, v))(x, s) JX_{t-s}^\varepsilon \varphi(X_{t-s}^\varepsilon) dx ds, \quad (3.15)$$

Using Hölder's inequality we obtain

$$\mathbb{E} \left| \int_0^t \int (\mathcal{R}_\varepsilon(V, v))(x, s) JX_{t-s}^\varepsilon \varphi(X_{t-s}^\varepsilon) dx ds \right|$$

$$\leq \left(\mathbb{E} \int_0^t \int |(\mathcal{R}_\epsilon(V, v))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^t \int |JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)|^2 dx ds \right)^{\frac{1}{2}}$$

From step 4 we deduce

$$\left(\mathbb{E} \int_0^t \int |(\mathcal{R}_\epsilon(V, v))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \rightarrow 0.$$

From estimation (3.12) we obtain

$$\begin{aligned} & \left(\mathbb{E} \int_0^t \int |JX_{t-s}^\epsilon \varphi(X_{t-s}^\epsilon)|^2 dx ds \right)^{\frac{1}{2}} \\ & \leq C \left(\int_0^t \int_{\mathbb{R}} |\varphi(x)|^2 dx ds \right)^{\frac{1}{2}} \leq C \int_{\mathbb{R}} |\varphi(x)|^2 dx. \end{aligned}$$

Passing to the limit in equation (3.15) we conclude that $V = 0$. Then we deduce that $u = 0$.

□

Acknowledgements

Christian Olivera C. O. is partially supported by CNPq through the grant 460713/2014-0 and FAPESP by the grants 2015/04723-2 and 2015/07278-0.

References

- [1] L. Ambrosio, (2004). *Transport equation and Cauchy problem for BV vector fields*, Invent. Math., 158, 227-260.
- [2] L. Ambrosio G. Crippa, (2014). *Continuity equations and ODE flows with non-smooth velocity*, Lecture Notes of a course given at Heriot Watt University, Edinburgh. Proceeding of the Royal Society of Edinburgh, Section A: Mathematics, 144, 1191-1244.

- [3] S. Attanasio and F. Flandoli.(2011) *Renormalized Solutions for Stochastic Transport Equations and the Regularization by Bilinear Multiplicative Noise*. Comm. in Partial Differential Equations, 36(8), 1455–1474.
- [4] L. Beck, F. Flandoli, M. Gubinelli and M. Maurelli, (2014) *Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness* . Preprint available on Arxiv: 1401-1530, .
- [5] S. Bianchini and A. Bressan, (2005) *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Annals of Mathematics, 161 , 223-342.
- [6] A. Bressan, G. Crasta and B. Piccoli, 2000 *Well-posedness of the Cauchy problem for 2×2 , systems of conservation laws*. Memoirs of the AMS, 146.
- [7] G.-Q. Chen, Q. Ding, and K. H. Karlsen,(2012). *On nonlinear stochastic balance laws*, Arch. Ration. Mech. Anal., 204,707-743,
- [8] A. Debussche, J. Vovelle,(2010). *Scalar conservation laws with stochastic forcing*, J. Funct. Anal. 259, 1014- 1042.
- [9] C.M. Dafermos, (2010) *Hyperbolic conservation laws in continuum physics*. Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 325. Springer-Verlag.
- [10] C. De Lellis, (2007) *Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio*, Bourbaki Seminar, Preprint, 1-26.
- [11] R. DiPerna and P.L. Lions, (1989) *Ordinary differential equations, transport theory and Sobolev spaces*. Invent. Math., 98, 511-547.
- [12] E. Fedrizzi and F. Flandoli, (2013) *Noise prevents singularities in linear transport equations*. Journal of Functional Analysis, 264, 1329-1354.
- [13] E. Fedrizzi, W. Neves, C. Olivera, (2014) *On a class of stochastic transport equations for L^2_{loc} vector fields*, to appears in the Annali della Scuola Normale Superiore di Pisa, Classe di Scienze., arXiv:1410.6631v2. .

- [14] J. Feng, D. Nualart(2008). Stochastic scalar conservation laws, J. Funct. Anal. 255, 313-373.
- [15] F. Flandoli, M. Gubinelli and E. Priola, (2010) *Well-posedness of the transport equation by stochastic perturbation*. Invent. Math., 180, 1-53.
- [16] M. Hofmanova(2016). *Scalar conservation laws with rough flux and stochastic forcing*, Stoch. PDE: Anal. Comp. 4 635-690.
- [17] K. T. Joseph,(1993) *A Riemann problem whose viscosity solutions contain δ -measures* , Asymptotic Analysis, 105-120.
- [18] S. Kruzhkov(1970). *First-order quasilinear equations with several space variables*, Mat. Sb. 123, 228-255. English transl. in Math. USSR Sb. 10 (1970), 217-273.
- [19] H. Kunita, (1990) *Stochastic flows and stochastic differential equations*, Cambridge University Press.
- [20] H. Kunita, (1982) *Stochastic differential equations and stochastic flows of diffeomorphisms*, Lectures Notes in Mathematics, Springer-Verlag, Berlin, 1097, 143-303.
- [21] H. Kunita,(1984) *First order stochastic partial differential equations*. In: Stochastic Analysis, Katata Kyoto, North-Holland Math. Library, 32, 249-269.
- [22] P.L. Lions, P. Benoit and P.E. Souganidis(2013). *Scalar conservation laws with rough (stochastic) fluxes* . Stochastic Partial Differential Equations: Analysis and Computations, 1 (4), 664-686.
- [23] P.L. Lions, P. Benoit and P.E. Souganidis(2014). *Scalar conservation laws with rough (stochastic) fluxes: the spatially dependent case*, Stochastic Partial Differential Equations: Analysis and Computations, 2, 517-538.
- [24] S.A. Mohammed, T.K. Nilssen, and F.N. Proske,(2015) *Sobolev Differentiable Stochastic Flows for SDE's with Singular Coefficients: Applications to the Transport Equation*, Annals of Probability, 43, 1535-1576.

- [25] David A.C. Mollinedo and C. Olivera.(2017) *Stochastic continuity equation with non-smooth velocity*, to appears Annali di Matematica Pura ed Applicata , Doi : 10.1007/s10231-017-0633-8 .
- [26] C. Olivera(2017). *Regularization by noise in one-dimensional continuity equation*, arXiv:1702.05971.
- .
- [27] D. Serre, (1999) *Systems of conservation laws 1-2*. Cambridge U. Press.
- [28] Dechun Tan, Tong Zhang and Yuxi Zheng, (1994) *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, Journal of Differential Equations, 112.